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IMPERFECTION SENSITIVITY OF FIBER MICRO-BUCKLING IN ELASTIC-NONLINEAR POLYMER-MATRIX COMPOSITES

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Abstract—Imperfection sensitivity of the fiber micro-buckling problem is a fundamental requirement for applying any fiber micro-buckling model for the prediction of composite compression strength. This paper is devoted to prove theoretically that fiber micro-buckling of elastic-nonlinear polymer-matrix composites is an imperfection sensitive problem. From the proof, it follows that compression strength can be predicted by a suitable model that includes fiber misalignment as the imperfection parameter. The conditions on the behavior of the constituents (fiber and matrix) for imperfection sensitivity are derived. A novel representation for the non-linear shear response of the composite is proposed and supported with experimental data. Also, arguments are presented to support the use of composite constitutive equations rather than using micro-mechanics, particularly when material non-linearity is involved. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Many models have been proposed trying to improve the prediction of compressive strength of composites by Rosen (1965). The literature is divided into two schools of thought: fiber micro-buckling models (Rosen, 1965, Wang, 1978, etc.) and kink band formation models (Budiansky and Fleck, 1993, etc.). In fiber micro-buckling models, it is assumed that the buckling of the fibers initiates a process that leads to the collapse of the material. Rosen's model (1965) has been refined with the addition of initial fiber misalignment and non-linear shear stiffness (Hahn and Williams, 1986, Yeh and Teply, 1988, Wisnom, 1990, Waas *et al.*, 1990, Chaudhuri, 1991, Yin, 1992, Lagoudas *et al.*, 1991, Highsmith *et al.*, 1992, Chung and Weitsman, 1994, 1995, Haberle and Matthews, 1994). The detrimental influence of fiber misalignment has been experimentally demonstrated (Yurgartis and Sternstein, 1992 and others.) In recent literature, the composite has been modeled using a representative volume element that includes a fiber with initial misalignment and a portion of matrix with some kind of material non-linearity.

Several difficulties have prevented fiber micro-buckling models from being accepted as the predominant physical model for compression failure of polymer matrix composites. From the stability point of view, the underlying assumption of recent models is that fiber micro-buckling of perfectly aligned fibers (Rosen's model) is an imperfection sensitive problem. Therefore, small amounts of imperfection (misalignment) could cause large reductions in the buckling load, thus the reduction of the compression strength with respect to Rosen's prediction. However, an existing theoretical study (Maewal, 1981) points out that fiber micro-buckling is not imperfection sensitive. The work to be presented here is devoted to prove theoretically that the problem of fiber micro-buckling is imperfection sensitive. The conditions for imperfection sensitivity derived in this paper shed light onto the class of materials for which the fiber micro-buckling models can be applied.

Most existing models use a two-dimensional array of unidirectional fibers supported by the matrix. Thus, the composite shear response is introduced through micro-mechanical relations, which are valid only in the context of linear elasticity. The inclusion of matrix shear non-linearity through micro-mechanical formulae, developed in the context of linear elasticity, is incorrect. The use of micro-mechanical models for the shear response is questionable even in the linear case because the *in situ* matrix shear modulus may be different than the neat resin value. This is because of the different polymer morphology obtained by curing the resin in the presence of coated fibers. In this investigation, the composite was modeled as a single element with no distinction between the fiber and matrix, similar to the earlier model of Wang (1978). This approach was adopted to reduce any inconsistencies that may result when using inadequate micro-mechanical formulae while including the non-linear shear response. The material properties were obtained directly from experimental results for the composite material. Although the fiber-matrix interface condition was shown to significantly affect the composite strength in the study by Madhukar and Drzal (1992), fiber-matrix failure appears to be an insignificant factor if an appropriate fiber coating is used (Yurgartis and Sternstein, 1992).

2. PERFECT SYSTEM

It is assumed in this work that compression failure is triggered by fiber micro-buckling in a shear model. The fiber micro-buckling load of the perfectly aligned composite is given by Rosen (1965) as

$$\sigma^c = \frac{G_m}{1 - V_f} \tag{1}$$

where G_m is the matrix initial stiffness, V_f is the fiber volume fraction, and G_{LT} is the initial shear stiffness of the composite approximated by the matrix dominated inverse rule of mixtures formula. Since eqn (1) predicts two to four times higher values of compression strength than experimental data, attempts have been made to improve Rosen's model. One possible explanation for the low compression strength is that the fiber-matrix system under axial load is imperfection sensitive. Rosen's prediction corresponds to a bifurcation point of the perfect system. Imperfection sensitivity can be studied on the perfect system. In this case, the associated post-buckling path for the fiber-matrix model but without imperfection is shown in Fig. 1, emerging from the bifurcation point. If the post-critical path of the

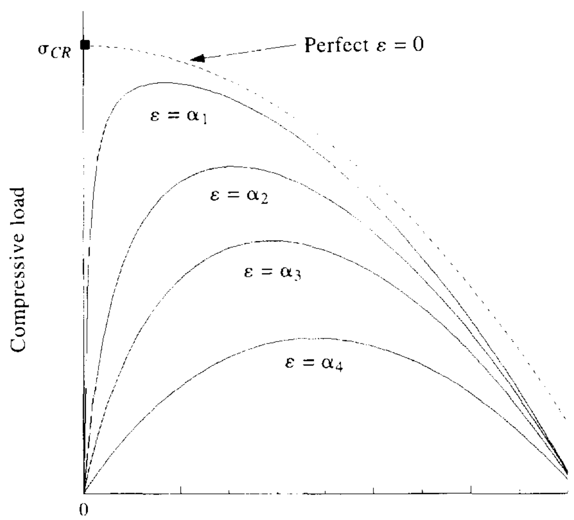

Fig. 1. Non-linear path showing (1) decreasing load in the post-buckling path of the perfect system, and (2) imperfection sensitivity in the imperfect system.

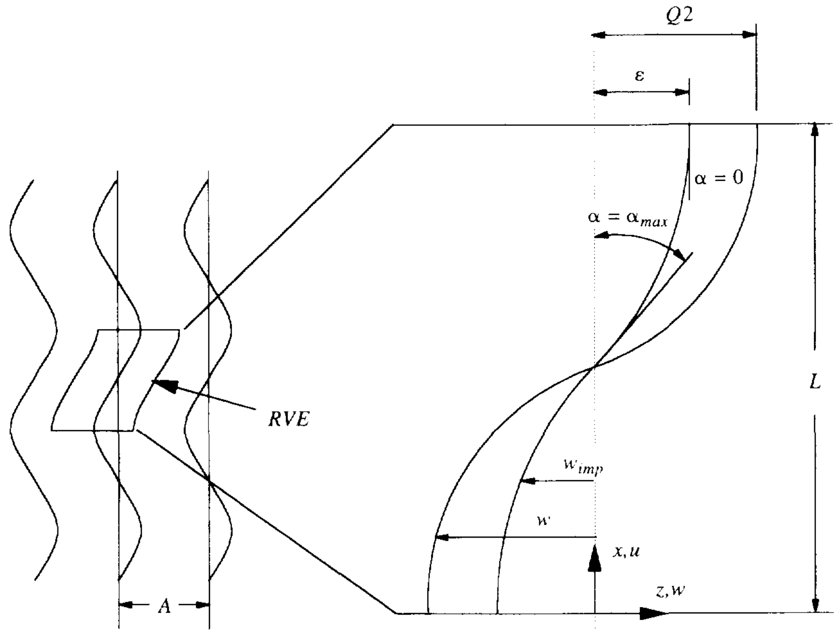

Fig. 2. Representative Volume Element (RVE) used for model formulation.

perfect system goes down, as in the figure, the imperfect system will have a limit point at a load much lower than the bifurcation load.

The nature of the post-critical path indicates whether or not the system is imperfection sensitive. Analysis of a perfect system (no misalignment) gives both the critical load and the post-buckling path. A model using the representative volume element (RVE) shown in Fig. 2, but with perfectly aligned fibers, is proposed. The model represents only the shear mode of micro-buckling, which is the dominant mode as shown by Rosen (1965).

The axial strain due to axial compression, retaining von Karman non-linear terms, is

$$\varepsilon_x = \varepsilon'_x + \varepsilon''_x$$

$$\varepsilon'_x = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\beta^2$$

$$\varepsilon''_x = -z\frac{\mathrm{d}\beta}{\mathrm{d}x}$$

$$\beta = \frac{\mathrm{d}w}{\mathrm{d}x}. \tag{2}$$

The shear strain in the periodic RVE (Fig. 2), caused by shear mode fiber deflection is

$$\gamma_{xz} = \frac{\mathrm{d}w}{\mathrm{d}x}. \tag{3}$$

All other strains are assumed negligible. Note that the shear deformation is with respect to the complete RVE and does not distinguish between fiber and matrix separately. The constitutive relations are assumed linear for the axial response

$$\sigma_x = E_L\varepsilon_x \tag{4}$$

where ε_x is given by eqn (2) and E_L is the longitudinal modulus of the composite element,

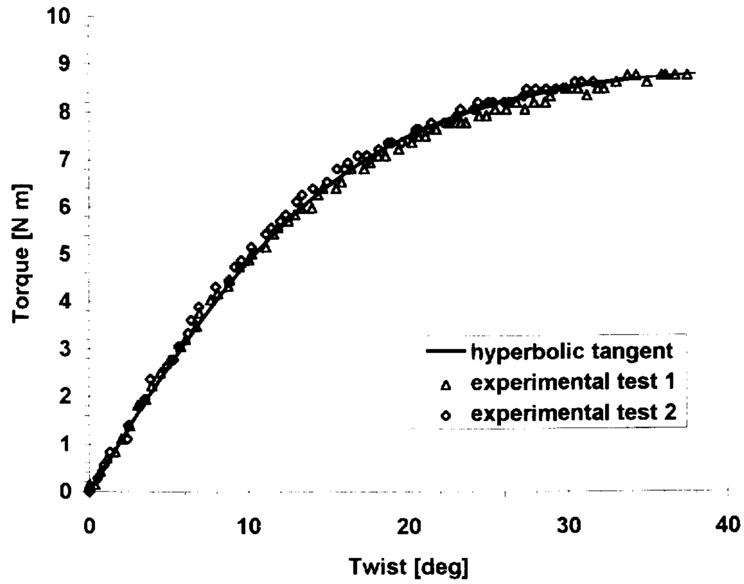

Fig. 3. Experimental shear response with hyperbolic curve-fit for glass/polyester composite ($G_{LT} = 4268$ MPa, $\tau_u = 38$ MPa).

to be determined experimentally or approximated by the rule-of-mixtures. The shear response is modeled with

$$\tau_{xz}(\gamma_{xz}) = \tau_u \tanh\left(\frac{G_{LT}}{\tau_u}\gamma_{xz}\right) \tag{5}$$

where G_{LT} and τ_u are the initial shear modulus and shear strength, respectively. Note that the shear response was assumed to be non-linear elastic based on experimental data from torsion tests (Tomblin, 1994). Steif (1990) used a hyperbolic model for the shear response of the composite, but its use was not supported by experimental measurements. In the current investigation, the use of the hyperbolic tangent relationship was experimentally verified using a torsion test on pultruded, fiber reinforced rods, and the constants G_{LT} and τ_u experimentally determined from the torque-twist data (Fig. 3).

Not only does the hyperbolic tangent fit well the initial portion of the data, but also represents the asymptotic behavior of the shear response in the large shear strain region. Unlike polynomial expansions, the hyperbolic tangent is antisymmetric with respect to the origin. In the stability analysis, a shear response which is not antisymmetric produces an asymmetric bifurcation point, which is physically unrealistic. Forcing the shear response to be antisymmetric by using abs(γ) makes the problem analytically untractable.

The total potential energy functional per unit depth is given by

$$V = \int_0^L \int_{-A/2}^{A/2} \left(\frac{1}{2}E_L\varepsilon_x^2 + \int_0^{\gamma_{xz}} \tau_{xz}(\gamma'_{xz})\,\mathrm{d}\gamma'_{xz}\right)\mathrm{d}z\,\mathrm{d}x + \int_0^L P\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \tag{6}$$

where L is the half-wavelength of the shear mode displacement function, A is the width of the RVE defined in Fig. 2, and P is the compressive load. The system was discretized using a Ritz approximation in the following form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = Q_1 \tag{7}$$

$$w = Q_2 \cos\left(\frac{\pi x}{L}\right) \tag{8}$$

where Q_1 and Q_2 are generalized coordinates (degrees of freedom) for the axial and transverse displacement, respectively. (Here du/dx is constant along x, and $w = 0$ at any state before buckling.) The generalized coordinate Q_2 represents the amplitude of the displacement of the fibers in the shear mode where L is the half-wavelength of the deflection. Hence, the shear strain is maximum at the position $L/2$ with respect the half-wavelength. The generalized coordinate Q_1 represents the axial strain.

Using eqns (2–6) in combination with eqns (7–8), the total potential energy of the system is

$$V = \frac{E_L A\pi^4 Q_2^4}{24L^3} + \frac{E_L ALQ_1^2}{2} + \frac{E_L A\pi^2 Q_1 Q_2^2}{4L} + \frac{E_L A^3\pi^4 Q_2^2}{48L^3} - \frac{A\pi L\tau_u^2 i}{G_{LT}} + \frac{2AL\tau_u^2}{G_{LT}}\ln\left[\exp\left(-\frac{\sqrt{2}G_{LT}\pi Q_2}{L\tau_u}\right)+1\right] - \frac{5AL\tau_u^2\ln(2)}{6G_{LT}} + \frac{A\pi\sqrt{2}\tau_u Q_2}{3} + \frac{6AL\tau_u^2}{G_{LT}}\ln\left[\exp\left(-2\frac{\pi G_{LT}Q_2}{L\tau_u}\right)+1\right] + \frac{A\pi\tau_u Q_2}{6} + PLQ_1 \tag{9}$$

where $A = D_f/V_f$ in which D_f is the diameter of the fibers and V_f is the fiber volume fraction.

The equilibrium path is found in terms of stress by setting to zero the first variation ($\delta V = 0$) of the potential energy, or for discrete systems $V_i = \partial V/\partial Q_i = 0$. Then

$$\sigma_{equ} = \frac{P}{A} = \frac{PV_f}{D_f} = -E_L Q_1; \quad Q_2 = 0. \tag{10}$$

The equilibrium path represents all the configurations, stable or not, which are in equilibrium. The path is described in terms of the generalized displacement Q_i and the load parameter $\Lambda = P/P_{cr}$, where P_{cr} is the load at the bifurcation point. Assuming the equilibrium path is stable in the undeformed and unloaded state $(Q_i, \Lambda) = (0, 0)$ and remains stable along the equilibrium path with increasing load parameter Λ until a critical point is reached, the distinct critical point (only one possible branch) is defined setting to zero the determinant of $\mathbf{V}_{ij} = \partial^2 V/\partial Q_i \partial Q_j$ (equivalent to the second variation for continuous systems). Subscripts of V indicate derivatives with respect to the generalized coordinates. The number of subscripts indicate the order of differentiation. Evaluating the determinant along the equilibrium path ($Q_2 = 0$), the derivatives are

$$V_{11} = \frac{E_L LD_f}{V_f}$$

$$V_{12} = V_{21} = 0$$

$$V_{22} = \frac{E_L D_f^3\pi^4}{24L^3 V_f^3} + \frac{E_L D_f\pi^2 Q_1}{2LV_f} + \frac{E_L D_f\pi^4 Q_2^2}{2L^3 V_f} + \frac{4D_f\pi^2 G_{LT}}{2Lv_f(1+S_2)} - \frac{4D_f\pi^2 G_{LT}}{3LV_f(S_2+S_2^2)^2} + \frac{2D_f\pi^2 G_{LT}}{3LV_f(1+S_1^2)} - \frac{2D_f\pi^2 G_{LT}}{3LV_f(S_1^2+S_1^4)}$$

$$S_1 = \exp\left(\frac{\pi G_{LT}Q_2}{L\tau_u}\right)$$

$$S_2 = \exp\left(\frac{\sqrt{2}\pi G_{LT}Q_2}{L\tau_u}\right). \tag{11}$$

Since $V_{12} = 0$, the matrix representing V_{ij} is diagonal. Then $\det(V_{ij}) = V_{11}V_{22}$. At a critical state det $(V_{ij})^c = V_{11}V_{22}|^c = 0$. Since V_{11} = constant, the critical point is found as $V_{22} = 0$. Solving for the critical point Q_1^c yields

$$Q_1^c = -\frac{G_{LT}}{E_L} - \frac{\pi^2 D_f^2}{12L^2 V_f^2} \tag{12}$$

which substituted into (10) yields the critical stress as

$$\sigma^c = G_{LT} + \frac{E_L \pi^2 D_f^2}{12L^2 V_f^2}. \tag{13}$$

Note that Rosen's model is recovered if $D_f/L \ll 1$. Now that the critical point has been established, the investigation turns to the stability of the critical point. Using the stability condition at the critical point $V_{ij}x_j|^c = 0$, the direction of the post-buckling path is obtained. The eigenvector $\mathbf{x}$ with components x_j was found as

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \tag{14}$$

The eigenvector points in the direction of the postcritical path and is orthogonal, at the critical point, to the fundamental path given by Q_i^c.

Upon obtaining the critical point along the equilibrium path, perturbation techniques are employed on the equilibrium equations to follow any branching or secondary paths. Choosing a generalized coordinate in the direction of instability (i.e., if the component Q_k is chosen as the perturbation parameter, the eigenvector component $x_k \neq 0$), the increments from the critical state may be described by

$$\begin{aligned} Q_i &= Q_i^c + q_i \\ \Lambda &= \Lambda^c + \lambda \end{aligned} \tag{15}$$

where

$$\begin{aligned} q_i &= q_i^{(1)} q_k + \frac{1}{2!} q_i^{(2)} q_k^2 + \frac{1}{3!} q_i^{(3)} q_k^3 + \ldots \\ \lambda &= \lambda^{(1)} q_k + \frac{1}{2!} \lambda^{(2)} q_k^2 + \frac{1}{3!} \lambda^{(3)} q_k^3 + \ldots \end{aligned} \tag{16}$$

in which q_k is the chosen perturbation parameter. A first derivative with respect to the perturbation parameter is denoted by $(\ )^{(1)}$, a second derivative is denoted by $(\ )^{(2)}$, etc. All derivatives are evaluated at the critical point. If q_1 is defined as the perturbation parameter ($s = q_1$), then the derivatives become $q_1^{(1)} = 1$, $q_1^{(2)} = 0$, $q_1^{(3)} = 0$, etc. An adequate perturbation parameter s is chosen to follow the secondary path. In the present case, a convenient choice is $s = Q_2$. Although Q_2 is unknown at this stage, the path is written in parametric form as a function of Q_2, leading to eqn (15).

The nature of critical point is determined by the contracted first order perturbation equation evaluated at the critical point (Thompson and Hunt, Section 5.2, 1973; Flores and Godoy, eqn 14, 1992). From the total potential energy for the system and using $(\ )'$ to denote a derivative with respect to the load parameter, we have

$$V_i' x_i = V_1' x_1 + V_2' x_2 = (L)(0) + (0)(1) = 0 \tag{17}$$

which indicates that the critical point is a bifurcation point (see Flores and Godoy, 1992).

The type of bifurcation point (asymmetric or symmetric) must now be determined. Using the total potential energy again, and following the general theory of elastic stability (Thompson and Hunt, 1973), we compute the coefficient of third order derivatives, defined as

$$C = V_{ijk} x_i x_j x_k |^c = V_{222} x_2 x_2 x_2 = 0. \tag{18}$$

A non-zero value of C would indicate an unstable critical point, while a zero value means that we have to investigate higher order derivatives to evaluate the stability of the state. From eqn (17) it follows that we are in the presence of a symmetric bifurcation, and its stability should be investigated with reference to the fourth order derivatives.

The nature of stability of the symmetric bifurcation point (either stable or unstable) is determined next using the perturbation equations. Choosing q_2 as the perturbation parameter for a symmetric bifurcation point, the following derivatives are obtained:

$$\begin{aligned} \lambda^{(1)} &= 0 \\ q_i^{(1)} &= x_i. \end{aligned} \tag{19}$$

Following the theory of elastic stability (Thompson and Hunt, 1973), the second order perturbation expansions in eqn (16) are

$$\begin{aligned} \lambda^{(2)} &= - \left. \frac{V_{ijkl} x_i x_j x_k x_l + 3 V_{ijk} x_i x_j z_k}{3 (V_{ijk} y_k + V'_{ij}) x_i x_j} \right|^c \\ q_i^{(2)} &= z_i + y_i \lambda^{(2)} \end{aligned} \tag{20}$$

where the auxiliary vectors $\mathbf{y}$ and $\mathbf{z}$ are defined (Flores and Godoy, 1992) as

$$\begin{aligned} V_{ij} y_j &= - V'_i \\ V_{ij} z_j &= - V_{ikl} x_k x_l. \end{aligned} \tag{21}$$

Since $\det(V_{ij})^c = 0$ and q_2 was chosen as the perturbation parameter, the components y_2 and z_2 are set arbitrarily equal to zero, in order to be able to solve the system (21), while being consistent with the second of eqns (20).

Using the above derivatives and the total potential energy eqn (9) for the defined system, the derivatives of the expansion are

$$\begin{aligned} q_1^{(1)} &= 0 \\ y_1 &= - \frac{V_f}{E_L D_f} \\ z_1 &= - \frac{1}{2} \frac{\pi^2}{L^2} \\ q_1^{(2)} &= - \frac{2}{9} \frac{\pi^2 \left(3E_L - 2 \frac{G_{LT}^3}{\tau_u^2} \right)}{E_L L^2} \\ \lambda^{(2)} &= \frac{1}{18} \frac{D_f \pi^2 \left(3E_L - 8 \frac{G_{LT}^3}{\tau_u^2} \right)}{V_f L^2}. \end{aligned} \tag{22}$$

Using the above results for the first and second order derivatives, it is possible to draw conclusions about the nature of stability of the critical point (Thompson and Hunt, 1973). Specifically, using the solution for $\lambda^{(2)}$, the secondary path can be approximated using the perturbation expansion. For most composite materials with a non-linear shear stress-strain response, the term containing G_{LT}^3/τ_u^2 will be significantly larger than the longitudinal modulus E_L. Hence, the derivative $\lambda^{(2)}$ will have a negative value indicating that the post-buckling, secondary path is unstable. A hyperbolic tangent representation of the composite shear response will always give an unstable secondary path, for the perfect system. In the theory of elastic stability (Thompson and Hunt, 1973) it is shown that if the perfect system has an unstable secondary path, the imperfect system is sensitive to imperfections. In this context, sensitivity means that the actual critical load of the imperfect system is lower than the critical load of the perfect system (bifurcation). Therefore, the composite is shown to be sensitive to imperfections in the form of fiber misalignment only if the shear response is nonlinear. Using eqns (15–16), the complete secondary path up to second order is given in terms of stress as

$$\sigma = \sigma^c + \frac{1}{2}\frac{\lambda^{(2)}}{A}q_2^2 \tag{23}$$

and substituting eqn (13) and the solution for $\lambda^{(2)}$ into eqn (23) yields

$$\sigma^c = G_{LT} + \frac{E_L \pi^2 D_f^2}{12L^2 V_f^2} + \frac{1}{36}\left(3E_L - 8\frac{G_{LT}^3}{\tau_u^2}\right)\left(\frac{\pi q_2}{L}\right)^2. \tag{24}$$

The second term in eqn (24) represents the contribution of the bending stiffness of the fiber and it is neglected with respect to the initial shear modulus G_{LT}, assuming that $D_f/L \ll 1$ for any value of V_f. This assumptions has also been used by Rosen (1965), Wang (1978), Hahn and Williams (1986), Yeh and Teply (1988). Hence, the secondary path becomes

$$\sigma^c = G_{LT} - \left(\frac{2}{9}\frac{G_{LT}^3}{\tau_u^2} - \frac{1}{12}E_L\right)\gamma^2 \tag{25}$$

where

$$\gamma = \frac{\pi q_2}{L} \tag{26}$$

represents the shearing strain (shear mode) at $x = L/2$ in the perfectly aligned fiber composite.

3. IMPERFECT SYSTEM

In the case of unstable, symmetric bifurcation points, small negative or positive imperfections change the behavior from a bifurcation point to a limit point case (Fig. 1). An unstable, symmetric bifurcation point of the perfect system indicates that the real system is imperfection sensitive. Hence, initial imperfections, in the form of initial fiber waviness or misalignment, are expected to significantly reduce the compression strength. An imperfection sensitivity analysis, based on total potential energy and the RVE shown in Fig. 2, is presented next. The assumed imperfection parameter ε is the amplitude of the initial fiber misalignment as shown in Fig. 2.

Beginning with the strain components for the shear mode type of failure, the axial strain due to an initial deflection of the fibers ε is

$$\varepsilon_{x_{imp}} = \varepsilon'_{x_{imp}} + \varepsilon''_{x_{imp}}$$

$$\varepsilon'_{x_{imp}} = \frac{1}{2}\left(\frac{\mathrm{d}w_{imp}}{\mathrm{d}x}\right)^2$$

$$\varepsilon''_{x_{imp}} = -z\frac{\mathrm{d}^2 w_{imp}}{\mathrm{d}x^2}. \tag{27}$$

The total axial strain is given by eqn (2). Next, we model the influence of the geometric imperfection as an initial strain field (Godoy, 1996). The true axial strain of the system becomes

$$\varepsilon_{x_{true}} = \varepsilon_x - \varepsilon_{x_{imp}}. \tag{28}$$

Similarly, the true shear strain is given by

$$\gamma_{xz_{true}} = \frac{\mathrm{d}w}{\mathrm{d}x} - \frac{\mathrm{d}w_{imp}}{\mathrm{d}x}. \tag{29}$$

Discretizing the system with

$$\frac{\mathrm{d}u}{\mathrm{d}x} = Q_1$$

$$w_{imp} = \varepsilon\cos\left(\frac{\pi x}{L}\right)$$

$$w = Q_2\cos\left(\frac{\pi x}{L}\right) \tag{30}$$

in which Q_1, Q_2 are generalized coordinates and ε is the initial amplitude of fiber deflection, the total potential energy for the imperfect system becomes

$$\begin{aligned} V = {} & \frac{E_L D_f \pi^4 \varepsilon^4}{24L^3 V_f} + \frac{E_L D_f \pi^2 Q_1 Q_2^2}{4LV_f} + \frac{E_L D_f^3 \pi\varepsilon^2}{48L^3 V_f^3} + \frac{E_L D_f^3 \pi^4 Q_2^2}{48L^3 V_f^3} - \frac{E_L D_f^3 \pi^4 \varepsilon Q_2}{24L^3 v_f^3} \\ & + \frac{E_L D_f Q_2^4 \pi^4}{24L^3 V_f} - \frac{E_L D_f \pi^4 \varepsilon^2 Q_2^2}{12L^3 V_f} - \frac{E_L D_f \pi^2 \varepsilon^2 Q_1}{4LV_f} - \frac{D_f \tau_u \sqrt{2}\pi\varepsilon}{3V_f} + \frac{D_f \tau_u \sqrt{2}\pi Q_2}{3V_f} \\ & - \frac{5LD_f \tau_u^2 \ln(2)}{6V_f G_{LT}} - \frac{LD_f \tau_u^2 \pi i}{V_f G_{LT}} - \frac{D_f \tau_u \pi\varepsilon}{6V_f} + \frac{D_f \tau_u Q_2}{6V_f} + \frac{E_L L D_f Q_1^2}{2V_f} \\ & + \frac{2D_f L\tau_u^2}{3V_f G_{LT}} \ln\left(\frac{S_4}{S_2}+1\right) + \frac{D_f L\tau_u^2}{6V_f G_{LT}} \ln\left(\frac{S_3^2}{S_1^2}+1\right) + pQ_1 L \end{aligned} \tag{31}$$

where S_1 and S_2 are defined in eqn (11) and S_3 and S_4 are given by

$$S_3 = \exp\left(\frac{\pi G_{LT}\varepsilon}{L\tau_u}\right)$$

$$S_4 = \exp\left(\frac{\sqrt{2}\pi G_{LT}\varepsilon}{L\tau_u}\right). \tag{32}$$

In order to obtain an estimate of imperfection sensitivity, the higher order derivatives of the expansion must be obtained using the perturbation equation (Thompson and Hunt, 1973). Note also that all derivatives are evaluated at the critical point of the perfect system in which $Q_2 = 0$, $Q_1 = Q_1^c$.

A first order approximation for the imperfection sensitivity is given by the 2/3 power law (Thompson and Hunt, Sect. 8.7, 1973)

$$\Lambda^M = \Lambda^C + \alpha(\varepsilon)^{2/3}$$
$$\alpha = \frac{\lambda^{M(2)}}{2}\left(\frac{6}{\varepsilon^{M(3)}}\right)^{2/3} \tag{33}$$

where Λ^M is the maximum of the load-deflection curve $\Lambda(Q)$ for any given imperfection ε, $\lambda^{M(2)}$ is the second derivative of the incremental load parameter λ with respect to the perturbation parameter, evaluated at the limit point. Choosing q_2 as the perturbation parameter, $\lambda^{M(2)}$ is given by

$$\lambda^{M(2)} = -\left.\frac{V_{ijkl}x_ix_jx_kx_l + 3V_{ijk}x_ix_jz_k}{V_{ijk}x_ix_jy_k + V'_{ij}x_ix_j}\right|^c = 3\lambda^{(2)}. \tag{34}$$

The coefficient $\varepsilon^M(3)$ is the third order derivative with respect to the perturbation parameter of the perturbation expansion of the imperfection ε,

$$\varepsilon^{M(3)} = \frac{-2\pi^2\left(3E_L - 8\dfrac{G_{LT}^3}{\tau_u^2}\right)}{E_LA^2\pi^2 + 12G_{LT}L^2}. \tag{35}$$

Using eqn (23), a first order approximation for the imperfection sensitivity (2/3 power law, Thompson and Hunt, 1973) is obtained in terms of stress as

$$\sigma^M = G_{LT} + \frac{E_L\pi^2A^2}{12L^2} + \frac{1}{12}\frac{\pi^{2/3}3^{2/3}}{L^2}\left(3E_L - 8\frac{G_{LT}^3}{\tau_u^2}\right)^{1/3}(E_LA^2\pi^2 + 12G_{LT}L^2)^{2/3}. \tag{36}$$

Neglecting the second term of eqn (36) as before and noting that $E_LA^2\pi^2 \ll 12G_{LT}L^2$ for most composite materials with realistic volume fractions, the imperfection sensitivity relation becomes

$$\sigma^M = G_{LT} + \frac{1}{2}2^{1/3}3^{1/3}G_{LT}^{2/3}\left(3E_L - 8\frac{G_{LT}^3}{\tau_u^2}\right)^{1/3}\alpha^{2/3} \tag{37}$$

where $\alpha = \pi\varepsilon/L$ represents the initial angle of misalignment for the imperfection system at a position $x = L/2$ (Fig. 2). As the misalignment increases, the maximum compressive stress decreases rapidly. However, it should be noted that perturbation methods are limited due to their slow convergence at high imperfection magnitudes since all perturbations are taken from the bifurcation point, in agreement with the argument presented by Steif (1990). Hence, a higher order perturbation expansion was developed by Tomblin (1994) in order to correlate the predictions with experiments. Because of the complexity of higher order perturbation expansions, a one degree-of-freedom, inextensional model was used. In this work, we have chosen to use the more general extensional model to obtain a more general proof of the imperfection sensitivity of the problem.

4. DISCUSSION

The bifurcation point for the perfect system (no misalignment) yields the critical stress (eqn 1) as found by Rosen (1965). More importantly, the system has been shown here to have an unstable post-buckling path. Therefore, compression strength is imperfection sensitive. Initial imperfections in the form of fiber misalignments are expected to destroy the bifurcation point and yield a maximum in the compressive stress lower than the bifurcation point. This result, presented here for the first time, is in contradiction to the findings of Maewal (1981) who predicted a stable post-buckling path. While Maewal used linear constitutive relations for axial and shear responses, imperfection sensitivity is made evident in this work by using the non-linear shear response.

The post-buckling trajectory of the perfect system is shown in Fig. 4. Using a linear shear response, a stable post-buckling path (shallow) is predicted, in agreement with the work of Maewal. In this case, initial imperfections in the form of fiber misalignment are not expected to affect the compressive strength. In contrast, when a non-linear shear response is used, the behavior changes from stable to unstable and initial imperfections are expected to decrease the maximum stress attainable in the composite. The fact that compression strength is sensitive to misalignment has been experimentally corroborated in the literature. This work presents the theoretical proof of imperfection sensitivity and presents the conditions on the material behavior for such a response. Based on this study we may speculate, for example, that misalignment may not be important in metal matrix composites (MMC) because of the linear shear stress-strain behavior of MMC.

The critical variables that influence compressive strength are the shear response and apparent fiber misalignment for the material system. For any given material system, the shear response should be fitted with a suitable equation, like eqn (5), using experimental data. Micro-mechanical models and the properties of the fibers and of neat resin cannot be used because existing micromechanical formulae are derived for linear behavior. Also, because the sensitivity of compression strength to the initial shear modulus G_{LT}, micro-mechanical models, which are not accurate to predict G_{LT}, are not used in this work.

All compressive strength models in the literature, including this one, are based on an ideal system of misaligned fibers, all having the same misalignment. Since all these models are extremely sensitive to fiber misalignment, the magnitude of fiber misalignment magnitude can be viewed as an empirical parameter. In other words, virtually any compressive

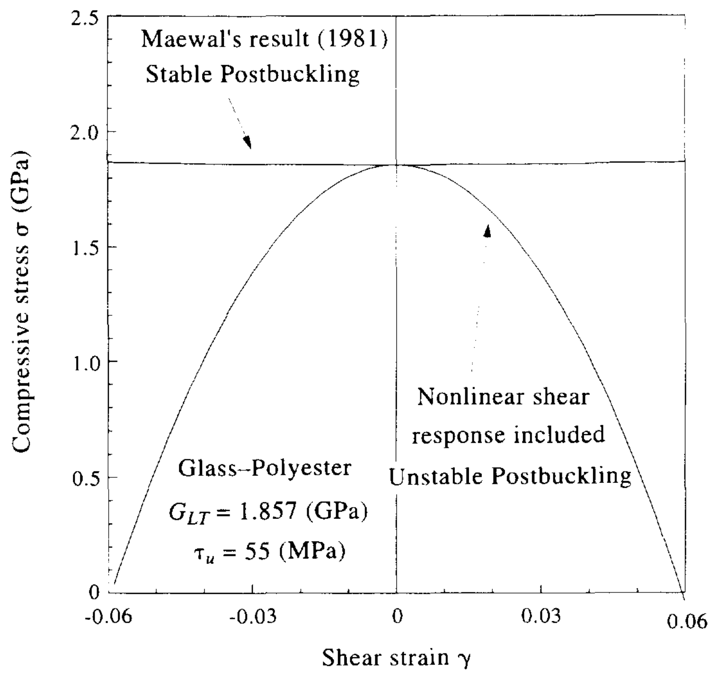

Fig. 4. Post-buckling trajectory of perfect system using linear and non-linear shear response.

strength model can be validated against experimental data using fiber misalignment as a fitting parameter, as long as the magnitude of misalignment is not actually measured. In fact, actual composites do not have a single misalignment but a distribution (see Yurgartis, 1987, Mrse and Piggott, 1990, Yurgartis and Sternstein, 1992, Rai *et al.*, 1992). Statistical proof that the fiber misalignment distribution can be approximated by a Gaussian distribution is presented by Tomblin (1994).

Simultaneous measurements of fiber misalignment, compressive strength, and shear response on the same material have been presented by Tomblin (1994). One objective of this manuscript is to analytically demonstrate that the postbuckling path of the perfect system is unstable symmetric. Thus, using only stability arguments, the behavior of the imperfect system is shown to be imperfection sensitive. Furthermore, it has been shown that the non-linear shear response eqn (5) must be included in order to obtain consistent results. Future work will be devoted to applying the stability model in conjunction with a statistical distribution of misalignment.

5. CONCLUSIONS

Fiber micro-buckling is theoretically proven to be imperfection sensitive due to an unstable, symmetric bifurcation point. Therefore, fiber misalignment is critical for the determination of compressive strength. Imperfection sensitivity cannot be detected unless the elastic non-linear shear response of the composite is incorporated in the formulation. The non-linearity in the shear stress-strain law were modeled by an equation that is antisymmetric with respect to the origin. Otherwise, unrealistic unsymmetric bifurcations are obtained. The initial fiber misalignment, initial shear modulus, and asymptotic shear response are critical variables in any compressive strength model. Because of limitations of space, the model is not developed here into a predictive tool for compression strength, which is presented elsewhere (Barbero and Tomblin, 1996). This paper presents the basis for such a tool. Specifically, higher order perturbation expansion needs to be used because the third order expansion used here is not accurate enough for large misalignment angles. Furthermore, the composite has a distribution of fiber misalignment and not a single value as modeled here (see Yurgartis, 1987, etc.). However, the limitations of the analysis presented here do not affect the stability of the post-critical path.

REFERENCES

Barbero, E. J. and Tomblin, J. (1996). A damage mechanics model for compression strength of composites. *Int. J. Solids Structures* **33,** 4379–4393.

Chaudhuri, R. A. (1991). Prediction of the compressive strength of thick-section advanced composite laminates. *J. Composite Mat.* **25,** 1244–1276.

Christensen, R. M. (1994). Material instability for fiber composites. *J. Appl. Mech.* **61,** 476–477.

Chung, I. and Weitsman, Y. (1994). A mechanics model for the compressive response of fiber reinforced composites. *Int. J. Solids Structures* **31,** 2519–2536.

Chung, I. and Weitsman, Y. (1995). On the buckling/kinking compressive failure of fibrous composites. *Int. J. Solids Structures* **32,** 2329–2344.

Flores, F. G. and Godoy, L. A. (1992). Elastic post-buckling analysis via finite element and perturbation techniques. Part I: formulation. *Int. J. Num. Meth. Engng* **33,** 1775–1794.

Godoy, L. A. (1996). *Thin-Walled Structures With Imperfections: Analysis and Behavior*, Elsevier, London.

Haberle, J. G. and Matthews, F. L. (1994). A micromechanics model for compressive failure of unidirectional fibre-reinforced plastics. *J. Composite Mat.* **28,** 1619.

Hahn, H. T. and Williams, J. G. (1986). Compression failure mechanisms in unidirectional composites. In *Composite Materials: Testing and Design (Seventh Conference)*, ASTM STP 893, American Society for Testing and Materials, Philadelphia, PA, pp. 115–139.

Highsmith, A. L., Davis, J. J. and Helms, K. L. E. (1992). The influence of fiber waviness on the compressive behavior of unidirectional continuous fiber composites. In *Composite Materials: Testing and Design* (Tenth Volume), ASTM STP 1120, (ed. Glenn C. Grimes), American Society for Testing and Materials, Philadelphia, pp. 20–36.

Lagoudas, D. C., Tadjbakhsh, I. and Fares, N. (1991). A new approach to microbuckling of fibrous composites. *J. Appl. Mech.* **58,** 473–479.

Madhukar, M. S. and Drzal, L. T. (1992). Fiber-matrix adhesion and its effect on composite mechanical properties. III. Longitudinal (0°) compressive properties of graphite/epoxy composites. *J. Comp. Mat.* **26,** 311–333.

Maewal, A. (1981). Post-buckling behavior of a periodically laminated medium in compression. *Int. J. Solids Structures* **17,** 335–344.

Mrse, A. and Piggott, M. R. (1990). Relation between fibre divagation and compressive properties of fibre composites. In *35th Int. SAMPE Symp.*, 2–5 April, pp. 2236–2244.
Rai, H. G., Rogers, C. W. and Crane, D. A. (1992). Mechanics of curved fiber composites. *J. Reinforced Plastics and Comp.* **11,** 552–566.
Rosen, B. W. (1965). *Fiber Composite Materials*, Ch. 3, American Society for Metals, Metals Park, OH.
Steif, P. S. (1990). A model for kinking in fiber composites—I. Fiber breakage via micro-buckling. *Int. J. Solids Structures* **26,** 549–561.
Sun, C. T. and Jun, A. W. (1993). Effect of matrix nonlinear behavior on the compressive strength of fiber composites. In *Mechanics of Thick Composites*, The 1st Joint meeting of the ASME-ASCE-SES: Meet'N'93, Charlottesville, VA, 6–9 June, pp. 91–105.
Thompson, J. M. T. and Hunt, G. W. (1973). *A General Theory of Elastic Stability*, John Wiley & Sons, New York, NY.
Tomblin, J. S. (1994). Compressive strength models for pultruded glass fiber reinforced composites. Ph.D. dissertation, West Virginia University, Morgantown, WV.
Waas, A. M., Babcock, C. D., Jr. and Knauss, W. G. (1990). A mechanical model for elastic fiber microbuckling. *J. Appl. Mech.* **57,** 138–148.
Wang, A. S. D. (1978). A non-linear microbuckling model predicting the compressive strength of unidirectional composites. In *ASME Winter Annual Meeting*, ASME Paper 78-WA Aero-1.
Wisnom, M. R. (1990). The effect of fibre misalignment on the compressive strength of unidirectional carbon fibre/epoxy. *Composites* **24,** 403–407.
Yeh, J. R. and Teply, J. L. (1988). Compressive response of kevlar/epoxy composites. *J. Comp. Mat.* **22,** 245–257.
Yin, W.-L. (1992). A new theory of kink band formation. *AIAA*-92-2552-*CP*, pp. 3028–3035.
Yurgartis, S. W. and Sternstein, S. S. (1992). Experiments to reveal the role of matrix properties and composite microstructure in longitudinal compression strength. *ASTM Symp. Compression Response of Comp. Structures*, Miami, FL, 16–17 November.
Yurgartis, S. W. (1987). Measurement of small angle fiber misalignments in continuous fiber composites. *Comp. Sci. and Tech.* **30,** 279–293.